MTS 101 - Analyse

Corrections : Petite Classe 3

Exercice 2

$$a)\ a_n = \frac{n^n}{n!}$$

$|\frac{a_{n+1}}{a_n}| = |\frac{(n+1)^{n+1}}{(n+1)!}\frac{n!}{n^n}| = |\frac{n+1}{n}|^n = (\frac{n+1}{n})^n$
$ln|\frac{a_{n+1}}{a_n}| = nln(1+\frac{1}{n}) = n(\frac{1}{n} - \frac{1}{2n^2} + o(\frac{1}{n^2})) = 1 - \frac{1}{2n} + o(\frac{1}{n})$
$ln|\frac{a_{n+1}}{a_n}| \to 1$ quand $n \to +\infty$
Donc par continuité de $exp(x)$ $|\frac{a_{n+1}}{a_n}| \to e$ quand $n \to +\infty$
Conclusion : Le rayon de convergence $R = \frac{1}{e}$

$$b)\ a_n = \frac{(-1)^n}{(n+1)^n}$$

$\sqrt[n]{|a_n|} = \frac{1}{n+1} \to 0$ quand $n \to +\infty$
Conclusion : Le rayon de convergence $R = +\infty$

$$c)\ a_n = \frac{(n+1)^p}{n!}$$

$|\frac{a_{n+1}}{a_n}| = (\frac{n+2}{n+1})^p\frac{1}{n+1} \to 0$ quand $n \to +\infty$
Conclusion : Le rayon de convergence $R = +\infty$

Exercice 3

$$a)\ f(z) = \frac{1}{1+z+z^2}$$

$f(z) = \frac{1}{1+z+z^2} = \frac{1-z}{(1+z+z^2)(1-z)} = \frac{1-z}{1-z^3}$

On sait que $\sum_{n=0}^{+\infty} z^n = \frac{1}{1-z}$ Donc $\sum_{n=0}^{+\infty} z^{3n} = \frac{1}{1-z^3}$

Conclusion : $f(z) = (1-z)\sum_{n=0}^{+\infty} z^{3n}$

$b)\ g(z) = \sin^2 z\, \cos z.$

$$\begin{array}{rcl} g(z) & = & \sin^2 z\, \cos z \\ & = & \sin z\,(\sin z\, \cos z) \\ & = & \sin z\,(\frac{1}{2}\, \sin 2z)\ \text{ car } sin(2z) = 2sinz\, cosz \\ & = & \frac{1}{2} \times \frac{1}{2}(\cos z - cos3z)\ \text{ car } sina\, sinb = \frac{cos(a-b)-cos(a+b)}{2} \\ & = & \frac{1}{4}(\cos z - cos3z) \end{array}$$

Puisque $cosz = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} z^{2n}$

alors $cos3z = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!}(3z)^{2n} = \sum_{n=0}^{+\infty} 3^{2n} \frac{(-1)^n}{(2n)!} z^{2n}$

Conclusion : $g(z) = \frac{1}{4}\sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!}(1-3^{2n})z^{2n}$